LIMITS

1. DEFINITION

The real number 'l' is called the limit of a function f(x) as 'x' tends to 'a' if for any $\in > 0$ and howsoever small there exists a real number d such that.

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - l| < \in$$

and then we write, $\lim_{x \to a} f(x) = \ell$

2. INDETERMINATE FORMS

The concept of limit was evolved to deal with indeterminate forms of some functions at some specific points. For example the function $f(x) = \dfrac{x^2 - 4}{x - 2}$ is ready to give its values at x = 0 , –1, 1, 3, $\sqrt{2}$ ..... etc.

$\because$ f(0) = 2, f(–1) = 1, f(1) = 3, $f(\sqrt{2}) = (2 + \sqrt{2})$.......etc.} but it fails to give its value at x = 2,

as $f(2) = \dfrac{2^2 - 4}{2 - 2} = \dfrac{0}{0}$ (known as an indeterminate form)

(Note, that $\dfrac{0}{0}$ is capable of assuming any value, that is why it is called 'indeterminate'), So in this case we find the limit, as follows.

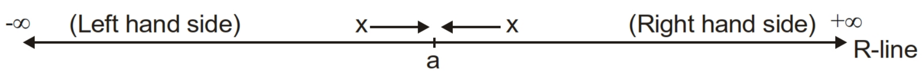
$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)\ (x+2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4 \quad ....(1)$$

Following indeterminate forms are in our course, namely

(i) $\dfrac{0}{0}$ (ii) $\dfrac{\infty}{\infty}$ (iii) $0 \times \infty$ (iv) $\infty - \infty$ (v) 1^{∞} (vi) ∞^0 (vii) 0^0

3. EVALUATION OF LEFT HAND LIMIT (LHL) AND RIGHT HAND LIMIT (RHL)

When 'x' approaches to 'a' i.e. when $x \to a$ then this variable 'x' on R–line can do this job in two ways one form the left hand side i.e. from negative side of the fixed point 'a' and second form the right hand side (i.e. from positive side) of 'a' as shown in the figure below



LHL is denoted by $\lim_{x \to a-0} f(x)$ or by $\lim_{x \to a^-} f(x)$ and similarly RHL by $\lim_{x \to a+0} f(x)$ or by $\lim_{x \to a^+} f(x)$ and are determined as (i.e., working rule is)

$$\text{LHL} = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) \text{ and } \text{RHL} = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h) \text{ ; where } h > 0$$

EXISTENCE OF LIMIT :

The limit of a function at a point exists if both left and right limits of the function at that point exist and are equal. Thus $\lim_{x \to a} f(x)$ exists $\Leftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

Note :– (i) Limit of a function at a point (if exists) is unique.

(ii) If at some point of a continuous function its value and limit both exist, then they are necessarily equal.

4. PROPERTIES OF LIMITS

· Let f and g be two real functions with domain D. We define four new functions $f \pm g, fg, \frac{f}{g}$, g ¹ 0 on domain (D) by setting

· $(f \pm g)(x) = f(x) \pm g(x)$ • $(fg)(x) = f(x)g(x)$

· $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, if $g(x) \neq 0$ for any $x \in D$.

Following are some results concerning the limits of these functions.

Let $\lim_{x \to a} f(x) = \ell$ and $\lim_{x \to a} g(x) = m$. If l and m exist.

(i) Sum and Difference Rule $\lim_{x \to a}(f \pm g)(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = \ell \pm m$

(ii) Product Rule $\lim_{x \to a}(fg)(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x) = \ell m$

(iii) Quotient Rule $\lim_{x \to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{\ell}{m}$, provided $m \neq 0$

(iv) Constant Multiple Rule $\lim_{x \to a} Kf(x) = K.\lim_{x \to a} f(x)$, where K is constant

(v) Modules Rule $\lim_{x \to a}|f(x)| = \left|\lim_{x \to a} f(x)\right| = |\ell|$

(vi) Power Rule $\lim_{x \to a}(f(x))^{g(x)} = \{\lim_{x \to a} f(x)\}^{\lim_{x \to a} g(x)}$

(vii) Composite Function Rule $\lim_{x \to a} fog(x) = f\left(\lim_{x \to a} g(x)\right) = f(m)$ In particular

(a) $\lim_{x \to a} \log f(x) = \log\left(\lim_{x \to a} f(x)\right) = \log \ell$

(b) $\lim_{x \to a} e^{f(x)} = e^{\lim_{x \to a} f(x)} = e^{\ell}$

(c) If $\lim_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \to a} \frac{1}{f(x)} = 0$

5. SOME STANDARD LIMITS

So from above Results

(a) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(b) $\lim_{x \to 0} \cos x = 1$

(c) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(d) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(e) $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a, (a > 0)$

(f) $\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$

(g) $\lim_{x\to 0}(1+x)^{1/x} = e$

(h) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$

(i) $\lim_{x\to a}\frac{x^m - a^m}{x-a} = ma^{m-1}$

(j) $\lim_{x\to a}\frac{x^m - a^m}{x^n - a^n} = \frac{m}{n}a^{m-n}$

6. METHODS OF EVALUATION OF LIMITS

(i) Substitution method

(ii) Factorization method

(iii) Rationalization or double rationalization

(iv) Expansion Method

(v) When $x \to \infty$

(vi) Simplification

(vii) L'Hospitals Rule

(viii) Sandwich Theorem

(ix) Evaluation of a limits of the form $0 \times \infty$, $\infty - \infty$, 0°, ∞°

(x) Evaluation of a limits of the form 1^∞

(i) Substitution method : In some cases limit of a function can be found by simple substitution if on substitution the function does not take indeterminate form.

(ii) Factorization method : If f(x) and g(x) are polynomials and $g(a) \neq 0$, then we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{f(a)}{g(a)}$$

(iii) Rationalization or double rationalization

(iv) Expansion Method : Using expansion of functions

If $x \to 0$ and there is atleast one function in the given expression which can be expanded, then we express Nr and Dr in the ascending powers of x and remove the common factor there.

The following expansions of some standard functions should be remembered:

(v) When $x \to \infty$

This type of problems are solved by the taking the highest power of the terms tending to infinity as common in numerator and denominator then take the limit of the function.

(vi) Simplification:

In this method the indeterminate form is removed by simplifying the expression.

(vii) L'Hospital's Rule:

(a) L'Hospital's Rule is applicable only when $\frac{f(x)}{g(x)}$ becomes of the form is $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

(b) If the form is not $\frac{0}{0}$ or $\frac{\infty}{\infty}$, simplify the given expression till it reduces to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then use L'Hospital's rule.

(c) For applying L'Hospital's rule differentiate the numerator and denominator separately.

If $f(a) = 0 = g(a)$ or $f(a) = \infty = g(a)$ then according to this rule

$$= \lim_{x\to a}\frac{f(x)}{g(x)}\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$$